Helmholtz's Theorems

Consider a **differential equation** of the following form:

$$g(t) = \frac{d\,f(t)}{dt}$$

where $g(t)$ is an **explicit** known function, and $f(t)$ is the **unknown** function that we seek.

For example, the differential equation :

$$3t^2 + t - 1 = \frac{d\,f(t)}{dt}$$

has a **solution**:

$$f(t) = t^3 + \frac{t^2}{2} - t + c$$

Thus, the **derivative** of $f(t)$ provides sufficient knowledge to determine the original function $f(t)$ (to within a constant).

An interesting question, therefore, is whether knowledge of the **divergence** and or **curl** of a vector field is **sufficient** to determine the original vector field.

For example, say we **don't** know the expression for vector field $\mathbf{A}(\bar{r})$, but we **do** know its divergence is some scalar function $g(\bar{r})$:

$$\nabla \cdot \mathbf{A}(\bar{r}) = g(\bar{r})$$

Can we, then, **determine** the vector field $\mathbf{A}(\bar{r})$? For example, can $\mathbf{A}(\bar{r})$ be determined from the expression:

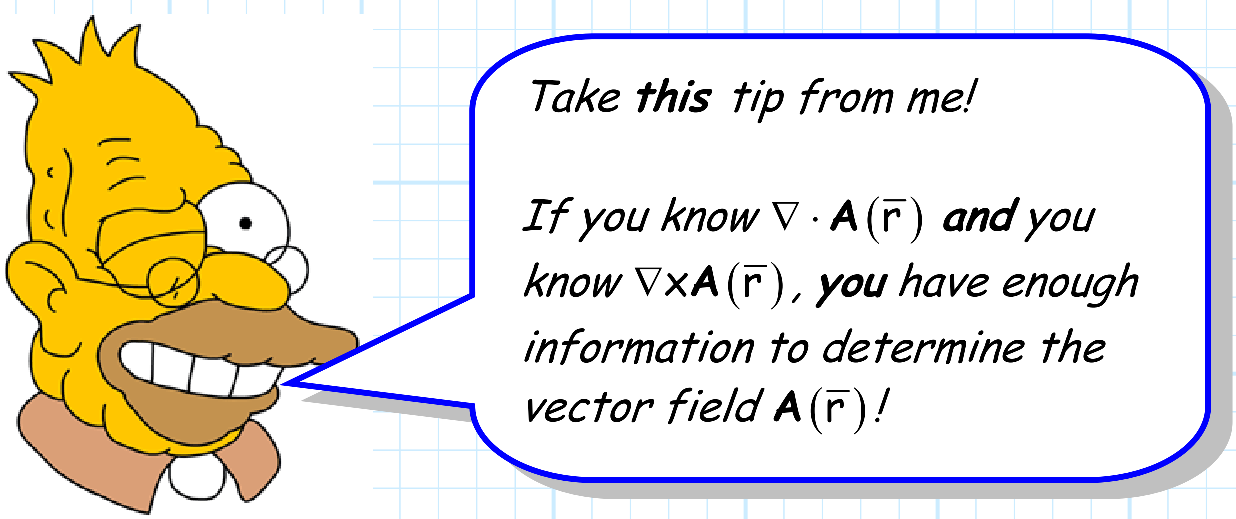
$$\nabla \cdot \mathbf{A}(\bar{r}) = x(y^2 - z^3) \quad ??$$

On the other hand, perhaps the knowledge of the **curl** is sufficient to find $\mathbf{A}(\bar{r})$, i.e.:

$$\nabla \times \mathbf{A}(\bar{r}) = \cos\frac{z\pi}{y}\,\hat{a}_x + (x^2 - 6)\,\hat{a}_y + e^{-\left(x/y\right)}\,\hat{a}_z$$

therefore $\mathbf{A}(\bar{r})$=????

It turns out that **neither** the knowledge of the divergence **nor** the knowledge of the curl **alone** is sufficient to determine a vector field. However, knowledge of **both** the curl and divergence of a vector field is sufficient!



Q: But **why** do we need knowledge of **both** the divergence and curl of a vector field in order to determine the vector field?

A: *I know the answer to that as well!*

*Its because **every** vector field can be written as the **sum** of a **conservative** field and a **solenoidal** field!*

That's correct! **Any** and **every** possible vector field $\mathbf{A}(\bar{r})$ can be expressed as the sum of a **conservative** field ($\mathbf{C}_A(\bar{r})$) and a **solenoidal** field ($\mathbf{S}_A(\bar{r})$) :

$$\mathbf{A}(\bar{r}) = \mathbf{C}_A(\bar{r}) + \mathbf{S}_A(\bar{r})$$

Note then if $\mathbf{C}_A(\bar{r}) = 0$, the vector field $\mathbf{A}(\bar{r}) = \mathbf{S}_A(\bar{r})$ is **solenoidal**. Likewise, if $\mathbf{S}_A(\bar{r}) = 0$ the vector field $\mathbf{A}(\bar{r}) = \mathbf{C}_A(\bar{r})$ is **conservative**.

Of course, if **neither** term is zero (i.e., $\mathbf{C}_A(\bar{r}) \neq 0$ and $\mathbf{S}_A(\bar{r}) \neq 0$), the vector field $\mathbf{A}(\bar{r})$ is **neither** conservative **nor** solenoidal!

Consider then what happens when we take the **divergence** of a vector field $\mathbf{A}(\bar{r})$:

$$
\begin{aligned}
\nabla \cdot \mathbf{A}(\bar{r}) &= \nabla \cdot \mathbf{C}_A(\bar{r}) + \nabla \cdot \mathbf{S}_A(\bar{r}) \\
&= \nabla \cdot \mathbf{C}_A(\bar{r}) + 0 \\
&= \nabla \cdot \mathbf{C}_A(\bar{r})
\end{aligned}
$$

Look what happened! Since the divergence of a solenoidal field is **zero**, the divergence of a general vector field $\mathbf{A}(\bar{r})$ **really** just tells us the divergence of its **conservative** component.

The divergence of a vector field tells us **nothing** about its solenoidal component $\mathbf{S}_A(\bar{r})$!

Thus, from $\nabla \cdot \mathbf{A}(\bar{r})$ we **can** determine $\mathbf{C}_A(\bar{r})$, but we haven't a **clue** about what $\mathbf{S}_A(\bar{r})$ is!

Likewise, the curl of $\mathbf{A}(\bar{r})$ is:

$$
\begin{aligned}
\nabla x \mathbf{A}(\bar{r}) &= \nabla x \mathbf{C}_A(\bar{r}) + \nabla x \mathbf{S}_A(\bar{r}) \\
&= 0 + \nabla x \mathbf{S}_A(\bar{r}) \\
&= \nabla x \mathbf{S}_A(\bar{r})
\end{aligned}
$$

Look what happened! Since the **curl** of a conservative field is **zero**, the curl of a general vector field $\mathbf{A}(\bar{r})$ **really** just tells us the curl of its **solenoidal** component.

The curl of a vector field tells us **nothing** about its conservative component $\mathbf{C}_A(\bar{r})$!

Thus, from $\nabla x \mathbf{A}(\bar{r})$ we can determine $\mathbf{S}_A(\bar{r})$, but we haven't a **clue** about what $\mathbf{C}_A(\bar{r})$ is!

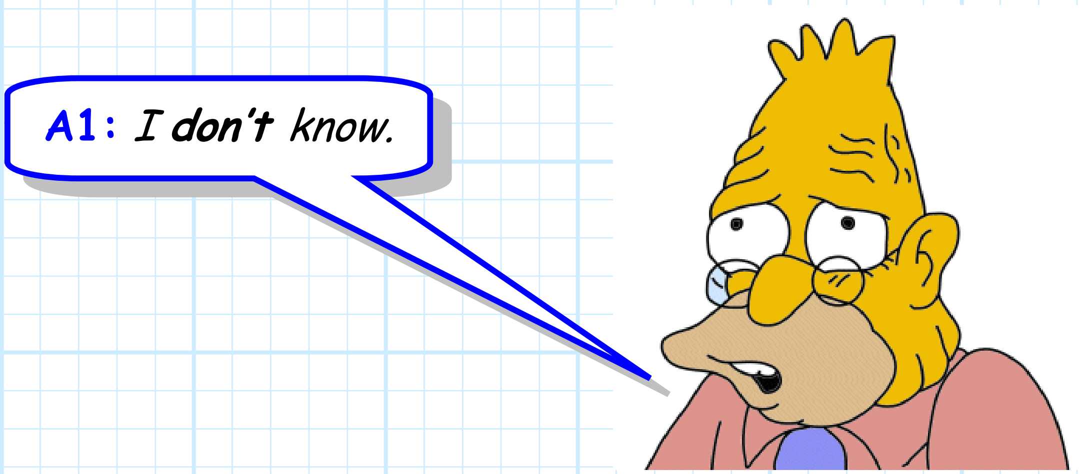
CONCLUSION: We require knowledge of **both** $\nabla \cdot \mathbf{A}(\bar{r})$ (for $\mathbf{C}_A(\bar{r})$) and $\nabla x \mathbf{A}(\bar{r})$ (for $\mathbf{S}_A(\bar{r})$) to determine the vector field $\mathbf{A}(\bar{r})$.

From a **physical** stand point, this makes perfect sense!

Recall that we determined the curl $\nabla x \mathbf{A}(\bar{r})$ identifies the **rotational sources** of vector field $\mathbf{A}(\bar{r})$, while the divergence $\nabla \cdot \mathbf{A}(\bar{r})$ identifies the **divergent** (or convergent) **sources**.

Once we know the **sources** of vector field $\mathbf{A}(\bar{r})$, we can of course **find** vector field $\mathbf{A}(\bar{r})$.

Q: Exactly **how** do we find $\mathbf{A}(\bar{r})$ from its sources ($\nabla \cdot \mathbf{A}(\bar{r})$ and $\nabla x \mathbf{A}(\bar{r})$) ??



A2: Note the **sources** of a vector field are determined from **derivative** operations (i.e., divergence and curl) on the vector field.

We can therefore conclude that a vector field $\mathbf{A}(\bar{r})$ can be determined from its sources with **integral** operations!

We'll learn **much more** about integrating sources later in the course!